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NUMERICAL SOLUTION OF A FIRST-ORDER CONSERVATION EQUATION BY A LEAST SQUARE METHOD

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SUMMARY

Least square methods have been frequently used to solve fluid mechanics problems. Their specific usefulness is emphasized for the solution of a first-order conservation equation. On the one hand, the least square formulation embeds the first-order problem into equivalent second-order problem, better adapted to discretization techniques due to symmetry and positive-definiteness of the associated matrix. On the other hand, the introduction of a least square functional is convenient for finite element applications.

This approach is applied to the model problem of the conservation of mass (the unknown is the density ρ) in a nozzle with a specified velocity field (u, v), possibly including jumps along lines simulating shock waves. This represent a preliminary study towards the solution of the steady Euler equations.

A finite difference and a finite element method are presented. The choice of the finite difference scheme and of a continuous finite element representation for the groups of variables $(\rho u, \rho v)$ is discussed in terms of conservation of mass flux. Results obtained with both methods are compared in two numerical tests with the same mesh system.

KEY WORDS First Order Equation Hyperbolic Conservation Equation Discontinuous Solutions Least Squares Finite Differences Finite Elements

INTRODUCTION

The solution by relaxation of the first-order equations or systems of conservation laws for subsonic and transonic flow of a perfect fluid, such as the steady Euler equations, is hindered by the difficult problem of the choice of the numerical scheme. This choice must ensure a proper domain of dependence of elliptic or hyperbolic type at each point, and a well-conditioned matrix for the associated algebraic system of difference equations thus allowing use of efficient inversion algorithms.

Least square methods have frequently been used to solve fluid mechanics problems (see, for example the excellent review by Eason[1]). Specifically, in the case of mixed-type equations, least square methods have proved very useful.[2,4] On the one hand, the least square formulation embeds the first-order problem into an equivalent second-order problem, better adapted to discretization techniques due to symmetry and positive-definiteness of the associated matrix. On the other hand, the introduction of a least square functional is convenient for finite element applications.

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is the density ρ) in a nozzle with a specified velocity field (u, v), possibly including jumps along lines simulating shock waves. This represents a preliminary study towards the solution of the steady Euler equations.

In the first part, the model problem and the associated least square formulation are presented.

A finite difference method is described in the second part. The second-order problem and the boundary conditions are written in a geometrically-oriented curvilinear co-ordinate system. The finite difference scheme is presented.

The third part concerns the finite element method. The choice of a continuous finite element for the groups of variables $(\rho u, \rho v)$ is discussed in terms of the conservation of mass flux.

In the last part, results of numerical tests are presented. They are concerned with a continuous velocity field and a discontinuous velocity field simulating a shock wave in the diverging part of the nozzle. The results of both methods are compared for the same mesh system. The latter case has also been solved by a pseudo-unsteady finite difference method. The relevance of the finite element approximation, illustrated by a one-dimensional normal shock problem, is contrasted with an approximation where, instead of the group (ρu), each variable, ρ and u, is represented separately by a continuous function.

1. FIRST-ORDER PROBLEM AND ASSOCIATED LEAST SQUARE FORMULATION

The nozzle flow domain is represented by an open set Ω of $\mathbb{R}^2$. Ω is simply-connected, bounded with a piecewise regular boundary Γ. The velocity field $\mathbf{q} = (u, v)$ is given on $\bar{\Omega}$ and satisfies the following hypothesis (Figure 1):

(i) $\mathbf{q}$ is regular, except on a finite number of regular curves Σ_i which divide Ω into disjoint subsets Ω_j and simulate shock waves;

(ii) the velocity modulus is strictly positive (no stagnation point in the flow field);

(iii) the streamlines do not coalesce and are not tangent to the curves Σ_i.

Let ρ be the density. ρ must satisfy the mass conservation equation:

$$A\rho = \operatorname{div} \rho\mathbf{q} = \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad \text{in } \Omega. \tag{1}$$

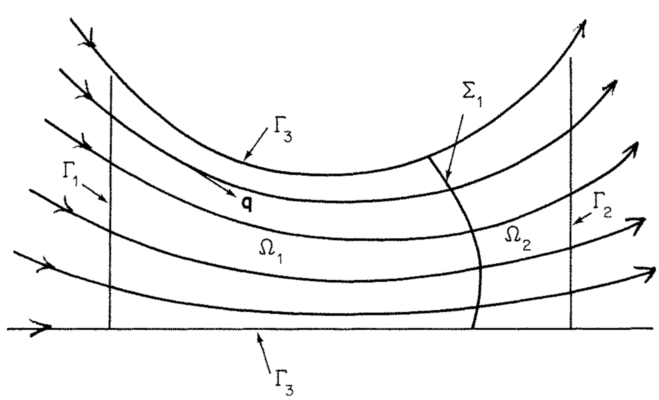

Figure 1. Domain Ω

Equation (1) is hyperbolic. The characteristic curves are the streamlines of the velocity field $\mathbf{q}$. A well-posed problem consists of specifying ρ on the part of the boundary of Ω where the flow is entering (upstream boundary).

Let

$$\Gamma_1 = \{(x, y) \in \Gamma \mid \mathbf{q} \cdot \mathbf{n} < 0\} \quad \text{(upstream boundary)}$$
$$\Gamma_2 = \{(x, y) \in \Gamma \mid \mathbf{q} \cdot \mathbf{n} > 0\} \quad \text{(downstream boundary)}$$
$$\Gamma_3 = \{(x, y) \in \Gamma \mid \mathbf{q} \cdot \mathbf{n} = 0\} \quad \text{(nozzle walls)}$$

where $\mathbf{n}$ is the external unit normal on Γ. Hence the upstream condition can be written:

$$\rho\mathbf{q} \cdot \mathbf{n} = g \quad \text{on } \Gamma_1. \tag{2}$$

On the curves Σ_i, ρ must satisfy a condition of 'Rankine–Hugoniot' type:

$$\langle \rho\mathbf{q} \cdot \mathbf{n}_i \rangle = 0 \quad \text{on } \Sigma_i \tag{3}$$

where $\langle a \rangle = a_2 - a_1$ represents the jump of a oriented by $\mathbf{n}_i$, the unit normal on Σ_i. Note that (3) is the expression of (1) in the sense of distributions.

The first-order problem can be summarized as:

Theorem: With the hypothesis (i), (ii) and (iii), the first-order problem (1), (2) (3) has a unique solution which is $C^1(\Omega_i)$ on each subset Ω_i.

This result is a consequence of the Cauchy–Kovaleska theorem[5]—the boundary condition on each Ω_i is obtained either from (2) or (3). The full, detailed proof can be found in Reference 6.

Let $L^2(\Omega)$ be the set of square-integrable functions in Ω. The least square formulation associated with the first order problem is:

Find ρ which satisfies (2) and minimizes

$$I(\tau) = \tfrac{1}{2} |\text{div } \tau\mathbf{q}|_{L^2(\Omega)} = \int_\Omega \tfrac{1}{2} (\text{div } \tau\mathbf{q})^2 \, dx \, dy.$$

We introduce the space of functions

$$V_\mathbf{q}(\Omega) = \{\tau \in L^2(\Omega) \mid \text{div } \tau\mathbf{q} \in L^2(\Omega)\}$$

$V_\mathbf{q}(\Omega)$ is a Hilbert space for the norm of the graph:

$$\|\tau\|^2_{V_\mathbf{q}(\Omega)} = |\tau|^2_{L^2(\Omega)} + |\text{div } \tau\mathbf{q}|^2_{L^2(\Omega)}.$$

The regularity of $\mathbf{q}$ assumes that $V_\mathbf{q}(\Omega)$ can be identified to a subspace of $H_{\text{div}}(\Omega) = \{\mathbf{p} \in L^2(\Omega)^2 \mid \text{div } \mathbf{p} \in L^2(\Omega)\}$ such that we can define a trace operator from $V_\mathbf{q}(\Omega)$ to $H^{-1/2}(\Gamma)$ by

$$\tau \mapsto \gamma_{\text{div}} \tau\mathbf{q}$$

where γ_{div} is the trace operator on $H \text{ div}(\Omega)$ (cf. Reference 8):

$$\gamma_{\text{div}} \, \mathbf{p} = \mathbf{p} \cdot \mathbf{n}.$$

This now gives a meaning to the functional I and the relations (2) and (3).

Thus the least square problem becomes:

$$I(\rho) = \inf_{\rho' \in V_1(\Omega)} I(\rho') = \inf_{\rho' \in V_1(\Omega)} \int_\Omega \tfrac{1}{2} \{\text{div } \rho'\mathbf{q}\}^2 \, dx \, dy \tag{4}$$

where $V_1(\Omega)=\{\rho' \in V_{\mathbf{q}}(\Omega) \mid \rho'\mathbf{q}\,.\,\mathbf{n}=g \text{ on } \Gamma_1\}$.

From the discretization point of view, in the finite element method the variational formulation of (4) is used and assumes the following form:

$$I(\rho)\,.\,\tau=\int_\Omega \operatorname{div}\rho\mathbf{q}\,.\operatorname{div}\tau\mathbf{q}\,\mathrm{d}x\,\mathrm{d}y=0, \qquad \forall\tau\in V_{\mathbf{q}}(\Omega)\mid \tau\mathbf{q}\,.\,\mathbf{n}=0 \text{ on } \Gamma_1. \tag{5}$$

In the finite difference method, the interpretation of (5) is used which, upon integration by parts, gives

$$\mathbf{q}\,.\,\mathbf{grad}\,(\operatorname{div}\rho\mathbf{q})=0 \quad \text{in } \Omega \tag{6.1}$$

$$\operatorname{div}\rho\mathbf{q}=0 \quad \text{on } \Gamma_2 \text{ and } \Gamma_3 \tag{6.2}$$

$$\rho\mathbf{q}\,.\,\mathbf{n}=g \quad \text{on } \Gamma_1 \tag{6.3}$$

Other, and more detailed, results are given in Reference 6.

2. FINITE DIFFERENCE METHOD

Equation (6.1) is written in a geometrically-oriented curvilinear co-ordinate system. In quasi-conservative form (the metric coefficients are brought in front of the derivatives) it becomes:

$$\begin{aligned}U\frac{\partial\xi}{\partial x}\frac{\partial^2\rho u}{\partial\xi^2}&+\left\{U\frac{\partial\eta}{\partial x}+V\frac{\partial\xi}{\partial x}\right\}\frac{\partial^2\rho u}{\partial\xi\,\partial\eta}+V\frac{\partial\eta}{\partial x}\frac{\partial^2\rho u}{\partial\eta^2}+U\frac{\partial\xi}{\partial y}\frac{\partial^2\rho v}{\partial\xi^2}\\&+\left\{U\frac{\partial\eta}{\partial y}+V\frac{\partial\xi}{\partial y}\right\}\frac{\partial^2\rho v}{\partial\xi\,\partial\eta}+V\frac{\partial\eta}{\partial y}\frac{\partial^2\rho v}{\partial\eta^2}+\left(u\frac{\partial^2\xi}{\partial x^2}+v\frac{\partial^2\xi}{\partial x\,\partial y}\right)\frac{\partial\rho u}{\partial\xi}+\left(u\frac{\partial^2\eta}{\partial x^2}+v\frac{\partial^2\eta}{\partial x\,\partial y}\right)\frac{\partial\rho u}{\partial\eta}\\&+\left(u\frac{\partial^2\xi}{\partial x\,\partial y}+v\frac{\partial^2\xi}{\partial y^2}\right)\frac{\partial\rho v}{\partial\xi}+\left(u\frac{\partial^2\eta}{\partial x\,\partial y}+v\frac{\partial^2\eta}{\partial y^2}\right)\frac{\partial\rho v}{\partial\eta}=0\end{aligned} \tag{7}$$

$U=\mathbf{q}\,.\,\mathbf{grad}\;\xi$, $V=\mathbf{q}\,.\,\mathbf{grad}\;\eta$ are the contravariant components of velocity. The metric coefficients $\frac{\partial\xi}{\partial x},\dots,\frac{\partial^2\xi}{\partial x^2}\dots$ are evaluated numerically.[7]

On Γ_2 and Γ_3 the condition (6.2) is transformed as:

$$\frac{\partial\xi}{\partial x}\frac{\partial\rho u}{\partial\xi}+\frac{\partial\eta}{\partial x}\frac{\partial\rho u}{\partial\eta}+\frac{\partial\xi}{\partial y}\frac{\partial\rho v}{\partial\xi}+\frac{\partial\eta}{\partial y}\frac{\partial\rho v}{\partial\eta}=0. \tag{8}$$

Equation (7) is discretized at each interior point of a rectangular (ξ,η) grid with indices (i,j), using a conservative, second-order accurate centred finite difference scheme:

$$\begin{aligned}\left(\frac{\partial\rho u}{\partial\xi}\right)_{i,j}&=\frac{(\rho u)_{i+1,\,j}-(\rho u)_{i-1,\,j}}{2\Delta\xi^2}\\\left(\frac{\partial^2\rho u}{\partial\xi^2}\right)_{i,j}&=\frac{(\rho u)_{i+1,\,j}-2(\rho u)_{i,\,j}+(\rho u)_{i-1,\,j}}{\Delta\xi^2}\\\left(\frac{\partial^2\rho u}{\partial\xi\,\partial\eta}\right)_{i,j}&=\frac{(\rho u)_{i+1,\,j+1}-(\rho u)_{i+1,\,j-1}-(\rho u)_{i-1,\,j+1}+(\rho u)_{i-1,\,j-1}}{4\Delta\xi\Delta\eta}\end{aligned} \tag{9}$$

with corresponding formulae for the other terms.The discretization of equation (7) yields a nine-point scheme everywhere.

On the boundary Γ_2 and Γ_3, the first-order equation (8) is discretized with a first-order accurate upwind scheme. It can be easily verified that in order to obtain a non-singular algebraic equation for the value $\rho_{i,j}$, the bias must be taken opposite to the flow direction. For example, in the situation sketched on Figure 2:

$$\left(\frac{\partial \rho u}{\partial \xi}\right)_{i,j} = \frac{(\rho u)_{i,j} - (\rho u)_{i-1,j}}{\Delta \xi}$$
$$\left(\frac{\partial \rho u}{\partial \eta}\right)_{i,j} = \frac{(\rho u)_{i,j} - (\rho u)_{i,j-1}}{\Delta \eta} \tag{10}$$

with corresponding formulae for the other terms.

A line relaxation method is employed to solve the algebraic system of equations. The relaxation factor is set close to one. Convergence is obtained in about 500 iterations on a 21×11 mesh system, which is comparable to what would be required to solve an elliptic problem with Neumann boundary conditions.

3. FINITE ELEMENT METHOD

The finite element discretization could be applied to the least square problem (4) as well as to the variational formulation (5). In both cases one must solve a linear system:

$$M\bar{\rho} = b$$

where M is an $N \times N$ matrix and $\bar{\rho}$ and b are N-dimensional vectors.

Let T_h be a finite element mesh with quadrangles K: Φ_K a finite set of degrees of freedom defined on each element K: Q_K a space of polynomials defined on the element K and of degree less than or equal to k in each direction ($k = 1$ in this application).

Let V_h be the set of continuous functions defined on Ω, the restriction of which on each element K of T_h is in Q_K. A function of V_h is uniquely defined by its values on Φ_K. V_h is a finite dimensional space. We shall denote by $\{\phi_i\}_{i=1,N}$ a base of V_h.

$\mathbb{V}_h = V_h \times V_h$ is thus and internal and converging approximation of $H_{\text{div}}(\Omega)$.[8]

Choice of the approximation

Let $(\rho u)_h$ and $(\rho v)_h$ be approximations of ρu and ρv respectively in V_h

$$\left.\begin{aligned}(\rho u)_h &= \sum_{i=1}^{N} (\rho u)_i \phi_i \\ (\rho v)_h &= \sum_{i=1}^{N} (\rho v)_i \phi_i.\end{aligned}\right\} \tag{11}$$

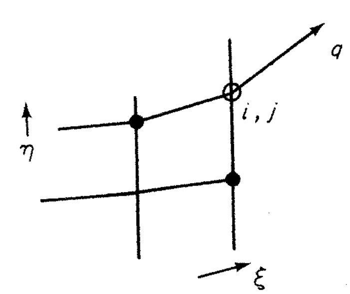

Figure 2. Discretization of equation (8) for $U \geq 0$, $V \geq 0$

Let $V_{\mathbf{q}h} = \{\rho_h$ such that $\rho_h \mathbf{q}_h = ((\rho u)_h, (\rho v)_h) \in \mathbb{V}_h\}$. Let us suppose that the intersection of the nodes of T_h with the curves Σ_i is empty. In this case one can write:

$$(\rho u)_i = \rho_i u_i$$
$$(\rho v)_i = \rho_i v_i.$$

In the discrete sense ρ_h is the vector $\bar{\rho} = \{\rho_i\}_{i=1,N}$. Replacing $V_{\mathbf{q}}$ by $V_{\mathbf{q}h}$ in (4) and (5) we get (i) the discrete least square formulation:

Find ρ_h in V_{1h} such that

$$I_h(\rho_h) = \operatorname*{Inf}_{\rho_h \in V_{1h}} I_h(\rho_h') = \operatorname*{Inf}_{\rho_h \in V_{1h}} \int_{\Omega_h} \frac{1}{2}\left(\frac{\partial(\rho' u)_h}{\partial x} + \frac{\partial(\rho' v)_h}{\partial y}\right)^2 \mathrm{d}x\,\mathrm{d}y \tag{12}$$

where $V_{1h} = \{\rho_h' \in V_{\mathbf{q}_h} \mid \rho_h' \mathbf{n}_h \cdot \mathbf{n} = g_h \text{ on } \Gamma_{1h}\}$.

(ii) the discrete variational formulation:

Find ρ_h in V_{1h} such that

$$I_h'(\rho_h) \cdot \tau_h = \int_{\Omega_h} \left\{\frac{\partial(\rho u)_h}{\partial x} + \frac{\partial(\rho v)_h}{\partial y}\right\}\left\{\frac{\partial(\tau u)_h}{\partial x} + \frac{\partial(\tau v)_h}{\partial y}\right\} \mathrm{d}x\,\mathrm{d}y = 0 \tag{13}$$
$$\forall \tau_h \in V_{\mathbf{q}_h} \mid (\tau \mathbf{q})_h \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{1h}$$

(12) and (13) yield the following algebraic system:

$$I_h(\rho_h') = {}^t\bar{\rho}' M \bar{\rho}'$$
$$I_h'(\rho_h) \cdot \tau_h = {}^t\bar{\rho} M \bar{\tau}$$

where the coefficients m_{ij} of M are:

$$m_{ij} = \int_{\Omega h} \left(u_i \frac{\partial \phi_i}{\partial x} + v_i \frac{\partial \phi_i}{\partial y}\right)\left(u_j \frac{\partial \phi_j}{\partial x} + v_j \frac{\partial \phi_j}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y.$$

Remarks on the approximation

The vector $\rho\mathbf{q}$ has been approximated by continuous functions. This is justified by an argument that $D(\Omega) \times D(\Omega)$ is dense in $H_{\mathrm{div}}(\Omega)$ (cf. Reference 8). It is not possible to construct an internal approximation of the space $V_{\mathbf{q}}(\Omega)$ of ρ with continuous functions in the case of a discontinuous velocity field $\mathbf{q}$.

Indeed, if we choose ρ, u and v separately in V_h we cannot find a solution which verifies

$$\frac{\partial \rho_h u_h}{\partial x} + \frac{\partial \rho_h v_h}{\partial y} = 0$$

with

$$\left.\begin{aligned} \rho_h u_h &= \sum_{i,j} \rho_i u_j \phi_i \phi_j \\ \rho_h v_h &= \sum_{i,j} \rho_i v_j \phi_i \phi_j. \end{aligned}\right\} \tag{14}$$

In other words, the least square solution is not a solution of the first-order system and the mass conservation is not ensured; this will be illustrated in the last part by a one-dimensional normal shock problem.

4. NUMERICAL TESTS

The domain Ω is bounded by the entrance and exit sections located at $x=-1$, $+1$ respectively, the axis of symmetry $y=0$ and the wall of the parabolic nozzle of equation $y=1-0{\cdot}2(1-x^2)$. The 21×11 mesh system for the methods of finite differences and finite elements is presented in Figure 3. It is composed of 200 elements which correspond to $N=220$ unknowns.

In the finite element procedure, the Q_1 functions are integrated on each quadrangle by a four-point Gaussian quadrature formula.

The numerical tests concern a continuous, divergence-free velocity field $\mathbf{q}_1$ and a discontinuous velicity field $\mathbf{q}_2$, simulating the presence of a shock wave in the diverging part of the nozzle.

The velocity field $\mathbf{q}_1$ is obtained by superposition of the velocity fields induced by two-point vortices distributed symmetrically with respect to the axis $y=0$ and having opposite strength:

$$\mathbf{q}_1=\begin{pmatrix}u_1\\ v_1\end{pmatrix}=\begin{pmatrix}\dfrac{x^2+4-y^2}{(x^2+4-y^2)^2+4x^2y^2}\\ \dfrac{2xy}{(x^2+4-y^2)^2+4x^2y^2}\end{pmatrix}.$$

The upstream condition (2) is

$$\rho_1=1 \quad \text{on } \Gamma_1$$

Thus the exact continuous solution is $\rho\equiv 1\ \forall(x,y)\in\Omega$. The results obtained by the finite difference and finite element methods are presented on Figures 4 and 5. It can be seen that the accuracy is good ($O(h^2)$) except for the finite difference method on the lines $j=1$, $j=11$ where the scheme is only first order accurate $O(h)$.

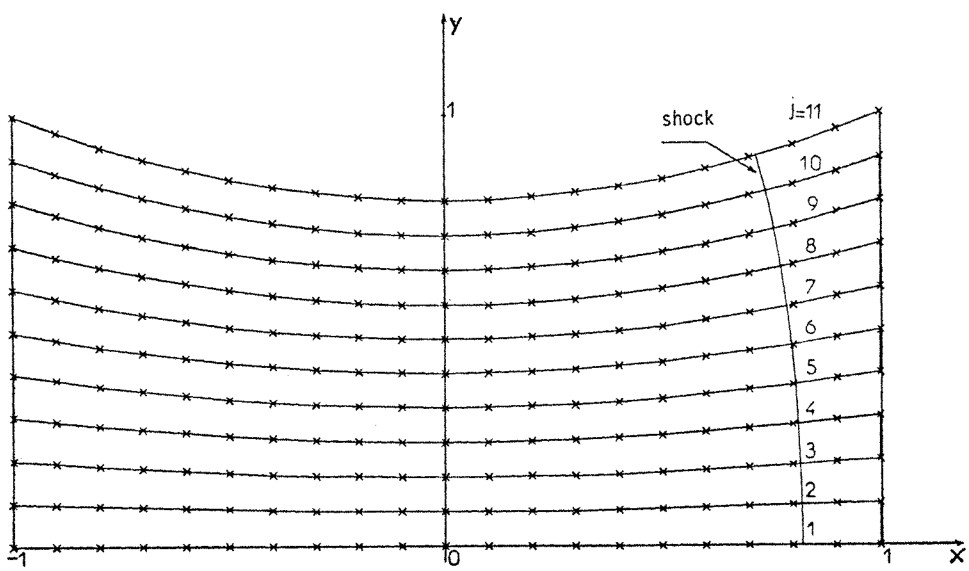

Figure 3. Mesh system

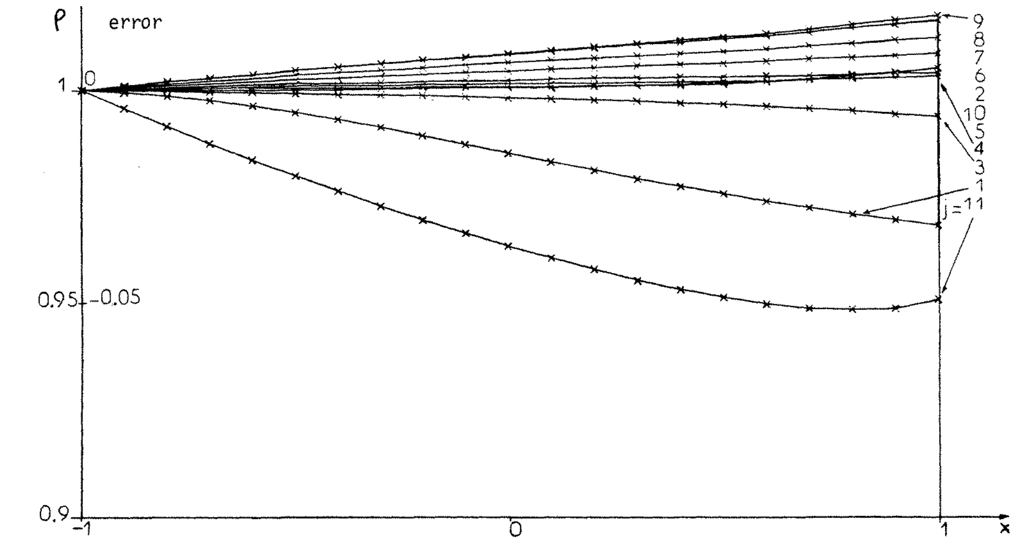

Figure 4. Continuous divergence-free velocity field finite difference method

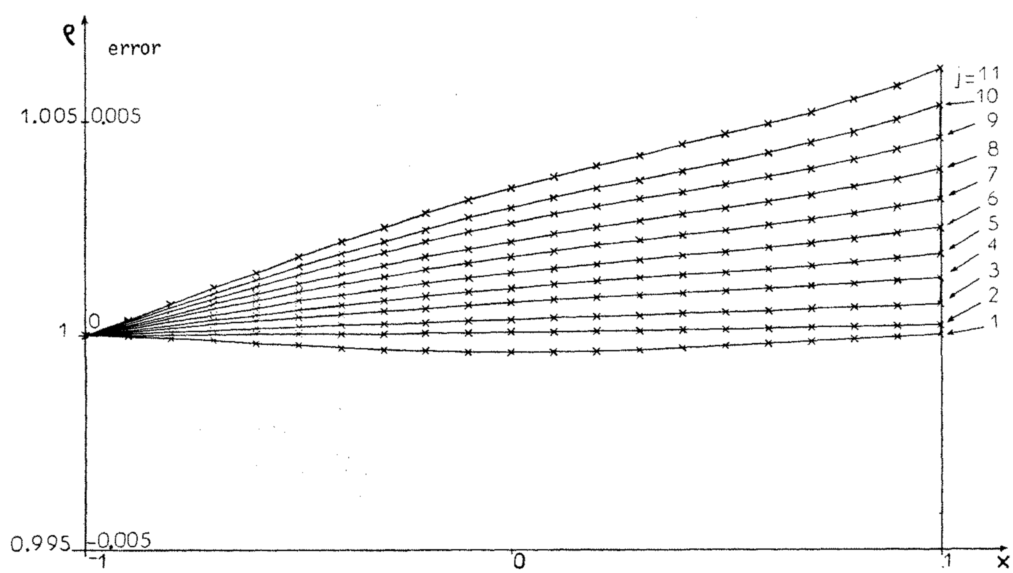

Figure 5. Continuous divergence-free velocity field finite element method

The velocity field $\mathbf{q}_2$ is defined as:

$$\mathbf{q}_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1+0\cdot5(1-x^2) \quad \text{if } x \geqslant x_{\text{shock}};\ 1+0\cdot5(1-x) \quad \text{if } x < x_{\text{shock}} \\ 0\cdot4xyu_2 \end{pmatrix}$$

with

$$x_{\text{shock}} = 0\cdot823532 - 0\cdot14y^2 \qquad \text{(Figure 3)}$$

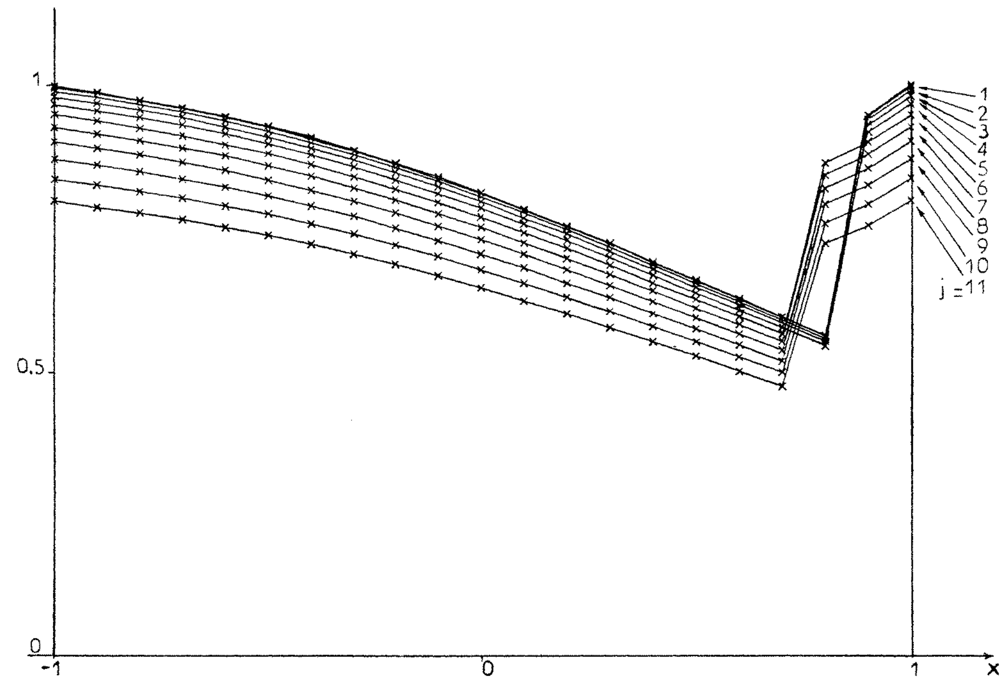

Figure 6. Discontinuous velocity field finite difference method

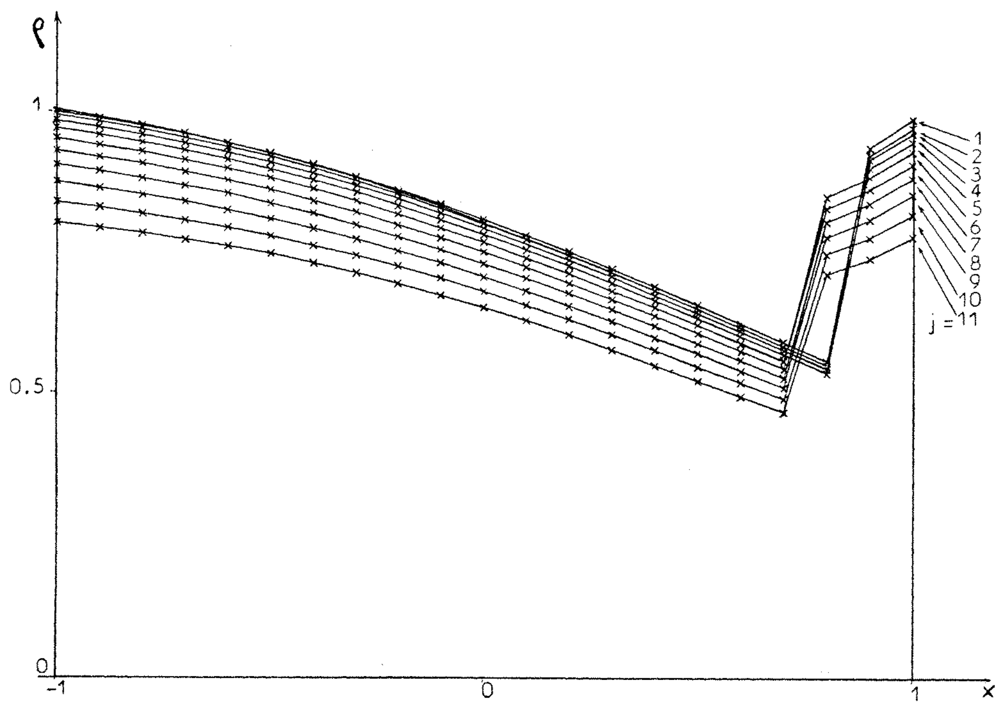

Figure 7. Discontinuous velocity field finite element method

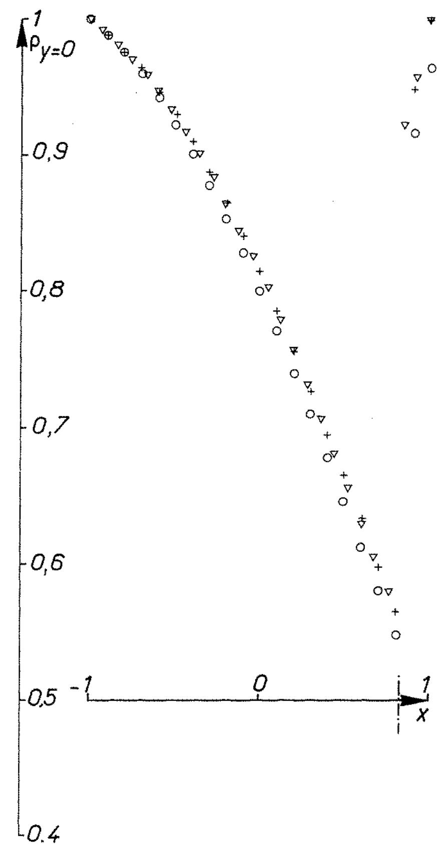

Figure 8. Density on the nozzle plane of symmetry. Comparisons between finite difference and finite element methods (∇ = pseudo unsteady finite difference method 26 × 11; + = 21 × 11 finite element method; ○ = 21 × 11 finite difference method)

The upstream condition (2) is

$$\rho_1(y) = 1 - 0{\cdot}2y^2 \quad \text{on } \Gamma_1.$$

Again the finite element result (Figure 7) yields a more accurate answer than the finite difference counterpart (Figure 6) when one considers the ratio of the exit to entrance flow rate. A loss of mass flux of 3 per cent is found in the finite difference solution as opposed to 0·1 per cent in the finite element solution.

In Figure 8 we present the density on the nozzle plane of symmetry computed by the finite difference method, the finite element method and a pseudo-unsteady second-order accurate finite difference method.[9] The latter and finite element methods are in excellent agreement although a different (slightly finer) mesh has been employed in the pseudo-unsteady method.

The final test case concerns a one-dimensional normal shock problem. It can be simulated by a horizontal, piecewise-constant velocity field in a channel with parallel walls. The number of elements in the x-direction is 10 and corresponds to a discretization step $h = 0{\cdot}2$. The results are presented in Table I for various values of the jump in u for the conservative (11) and non-conservative (14) finite element approximations. They illustrate the remark made earlier.

Table I. One-dimensional normal shock problem

	$\sum \rho_i u_i \phi_i$				$\sum_{ij} \rho_i u_j \phi_i \phi_j$			
JUMP $\langle u \rangle$	Iteration count	Error	Cost	Relative flux error	It. count	Error	Cost	Relative flux error
0·00	1	10^{-7}	10^{-6}	0	1	10^{-7}	10^{-5}	0
0·20	16	8.10^{-4}	$1{\cdot}2.10^{-5}$	0·004	16	8.10^{-4}	$1{\cdot}0.10^{-3}$	0·007
0·40	18	9.10^{-4}	$3{\cdot}0.10^{-5}$	0·005	18	8.10^{-4}	$1{\cdot}5.10^{-2}$	0·030
0·60	20	8.10^{-4}	$3{\cdot}5.10^{-5}$	0·005	18	9.10^{-4}	$7{\cdot}6.10^{-2}$	0·100
0·80	21	8.10^{-4}	$3{\cdot}2.10^{-5}$	0·005	18	8.10^{-4}	$2{\cdot}1.10^{-1}$	0·299
1·00	21	9.10^{-4}	$5{\cdot}6.10^{-5}$	0·006	17	9.10^{-4}	$4{\cdot}7.10^{-1}$	0·360
1·20	22	8.10^{-4}	$5{\cdot}6.10^{-5}$	0·005	16	9.10^{-4}	$3{\cdot}5.10^{-1}$	0·50
1·40	22	9.10^{-4}	$7{\cdot}5.10^{-5}$	0·006	15	9.10^{-4}	1·36	1·18
1·60	22	9.10^{-4}	$1{\cdot}1.10^{-4}$	0·007	15	8.10^{-4}	2·00	1·02
1·80	23	8.10^{-4}	$1{\cdot}0.10^{-4}$	0·006	14	8.10^{-4}	2·77	1·3
2·00	23	8.10^{-4}	$1{\cdot}1.10^{-4}$	0·007	14	8.10^{-4}	3·64	3·52
4·00	24	8.10^{-4}	$3{\cdot}3.10^{-4}$	0·001	14	8.10^{-4}	17·51	4
6·00	24	9.10^{-4}	$6{\cdot}4.10^{-4}$		14	8.10^{-4}	39·60	6
8·00	25	8.10^{-4}	$1{\cdot}0.10^{-3}$		14	9.10^{-4}	69·67	8
10·0	25	8.10^{-4}	$3{\cdot}8.10^{-3}$	0·002	14	9.10^{-4}	100·0	10·3

In conclusion, it has been shown that the least square formulation can prove very useful for the numerical solution of first-order conservation equations. It can be employed with the finite difference and finite element types of discretization. A proper scheme consists in approximating the group of variables ρu, ρv in the finite difference as well as finite element methods in the presence of discontinuities. The implementation of the finite element method has been done according to the directive proposed in Reference 10 and does not present any difficulty, more so when the elements are continuous and of low degree.

REFERENCES

1. E. D. Eason, 'A review of least-squares methods for solving partial differential equations', *Int. J. num. Meth. Engng*, **10,** 1021–1046 (1976).
2. J. J. Chattot, 'Relaxation approach to the steady Euler equations in transonic flow', *AIAA paper, No.* 77–636—*3rd Conf. Computations Fluid Dynamics*, Albuquerque, N.M., U.S.A., 21–22 (1977) June.
3. J. J. Chattot, J. Guiu-Roux and J. Laminie, 'Résolution numérique d'une équation de conservation par une approche variationnelle', *Proc.* 6*th Int. Conf. Num. Meth in Fluid Dyn.*, Tbilissi, U.S.S.R., 20–25 June (1978). Published by Springer-Verlag, 1979.
4. J. J. Chattot, J. Guiu-Roux and J. Lamine, 'Finite element calculation of steady transonic flow in nozzles using primary variables', *Proc. 7th Int. Conf. Num. Meth. in Fluid Dyn.*, Stanford/NASA Ames, 23–27 June (1980), Lecture Notes in Physics, W. C. Reynolds, R. W. MacCormack Edit. no. 141, Springer-Verlag, 1981.
5. R. Courant and D. Hilbert, Methods of Mathematical Physics, *Vol. II: Partial Differential Equations*, Interscience, 1962.
6. J. Guiu-Roux and J. Lamine, To be published.
7. J. J. Chattot, C. Coulombeix and C. Tomé, 'Calculs d'écoulements transsoniques autour d'ailes' *La Recherche Aérospatiale No.* 1978–4, pp. 143–159. English Translation ESA TT 561 (1979).
8. R. Temam, Navier–Stokes Equations, North Holland, 1977.
9. J. P. Veuillot and H. Viviand, 'A pseudo-unsteady method for the computation of transonic potential flows', *AIAA paper No.* 78–1150 (1978).
10. G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, 1973.